LETTER TO THE EDITOR

Topological properties of percolation clusters

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Abstract. Percolation properties, including the total cluster mass S, the shell mass B, and the linear geometrical size R, are studied as a function of the topological 'chemical distance' parameter L. All critical exponents are shown to be related to an apparently new exponent $\tilde{\nu}$, defined by $R \sim L^{\tilde{\nu}}$. Critical exponents are calculated exactly for percolation clusters on the Cayley tree (a model for 6D percolation), for which $S \sim L^2$, $B \sim L$, and $R^2 \sim L$. For diffusion on such clusters one finds that $L^3 \sim t$. Numerical estimates of the exponents are obtained for other dimensions. A conjecture which relates $\tilde{\nu}$ to β and ν is discussed.

The structure of percolation clusters recently has been a subject of intensive investigation (Stanley 1977, Mandelbrot 1982, Gefen *et al* 1981, Alexander and Orbach 1982, Coniglio 1982, Ben-Avraham and Havlin 1982, Stanley and Coniglio 1983). Characteristic exponents such as the fractal dimensionality d_f and the fracton dimensionality $\bar{d}$ have been given particular attention. The first of these exponents, d_f, relates the mass of a cluster, S to its *spatial* (i.e., geometrical) linear size, R, as $S \sim R^{d_f}$. The exponent $\bar{d}$, in contrast, characterises a *topological* property of the cluster, namely, the number of distinct sites in the cluster, S_t, visited by a random walk up to time t, which is given by $S_t \sim t^{\bar{d}/2}$. In this letter we investigate other topological quantities which characterise percolation clusters.

We first introduce the notion of the 'chemical distance' L. In figure 1 we show an example of part of an infinite percolation cluster at criticality, grown on a triangular lattice by the method of Leath (1976) and Alexandrowicz (1980). Occupied sites of the cluster are designated by letters. Dots and blank spaces signify unoccupied sites. The 'chemical distance' between two occupied sites will be defined as follows. Suppose one arbitrarily chooses as a reference point the site designated by the asterisk in figure 1. The occupied nearest neighbours of that site are designated by the letter 'B', and represent a 'shell' lying at chemical distance $L = 1$ from the reference site. The next-nearest-neighbour sites are designated by 'C' and constitute a shell at chemical distance $L = 2$, and so forth. The following quantities are functions of L: $B(L)$ is the number of occupied sites in the Lth shell, $S(L)$ is the total number of such sites included in a sub-cluster bounded by the Lth shell, and $R(L)$ is the radius of gyration of that sub-cluster. $B(L)$ and $S(L)$ are topological quantities since they do not depend directly on the spatial distribution of the sites.

Averages of the quantities $B(L)$, $S(L)$, and $R(L)$ at the percolation threshold p_c can be related as follows. Let us consider *only* the ensemble of clusters which contain

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        ..A  ZZ.            A
        Y   .U .HM KJ H EDC.CD.GG       .Q R
        X ..U T HMLKJIHGF CBB D.FFG      .PQR
        XW UTS  HM ..  G .BA.CC E.H  HOPQR
        XWV TSR O ..   HHI  Z.BCD I .LHN  R.    ..
     A.   .Y. UTSRQP.         YYZABCD  KK H      ..  .Q
     Z..Z.XWV TS QQ.           X AB D   J Q.    .RQ P
     ZY.YZY    SR.R U.        .WW BC E HI   .  S.PO...
     A XX YZ  UTSSSST.         V  C  FGH      RQ HHO.
     XWW.XY N. .T T U  XW  UUU  ..HIJK   .SR..H  ..
   AZYXWVVWX HML. ..VVW.Y VU.TT.U   I .LM .X T..H L ..G..
   .ZYX U.  NHLKJIIJ.W .X V.T.SSTU.  .HHN.W.U .MLK.HGFFG.
   AZ Y T R  H. K H. F WVUTSRR..     H....V.  ..JIH.E.G H.
     YX.TSRQ..  .HGFF E X..UT QQ....     O....UV.  ...D FGH.
   AZ W.TS PO LKJI .E.D Y Y. PPP        .RST..     .BCDEF.
     ..WVUT . HM ..JJ.EDC.Z.YXWV..OO P..TSRQ QRS.U    ZABC..
     .YX VU X OHHQ  K .CBAAZ  W.H NOPO  .RQPPQ..TT.  .XY.BCD.  .LH.
      . .VWX .OP.LLH. B..  X MM . HH ..DP. SST  W.ZABCD...K.L
      .W X  P.... N.C.  R .Y LLHH H.HH H..P .RRS UV.A..DEFG.JKL.
     AZYX  Y. QR.    O.   .Q  LK  .L. LH.H OPQQ.TTU B .FE FGHIJ.HN.
   A.A   YY.Z   R.Q  ..   .RQP H.J.JKL.K.ML.H .P .S. CCD FF ..IJK.H. BB
   Z ZA AZZZAA .RQPP...   .RQPQH .II.LLKJ..K H.O QR. .DEF G.. J.L. .A.
     .YY  AAA    .O.C. .TSR...  H H.KJIIJ LHHQP.  .EE GH H. .HH . ZA.
     .YXX    HHH.B    .S. .G .K.HI.K HQ.Q. .FF    H O H WWXYZ.B
      ..WWH.    .LLH BAZ..VUTT.    FFF GHIJ  ..G GHIJKLHHQOO.VWXYZA.
     YXWWV. .KK. C ZYXW.UU    .FEEE F. J ..P. . JK..N P .U. ZAB
     YXWVU    ..J  D A .X.V    EDD..E KJ.HH O....  .K .V DEF.STU  AB
     ZYX. TS...H. .JI F .B.AZY.WW     EDCCCDE..I.HHN.VU. W..L R.U. QRSTU.
      Y U R OHH.K.IH EDC..AZ X .      DCBBC FGHI L.O.UTT.V H..Q TSRR U.
      .Z.Z U. QP HLKJIHGFE .. A  .    .B#B  H JKL P .STUV.NOP U.55 VV ..B
     AZY  TSR OH LK..H.F  HH.BB.U.    .DC  C .I KKL OPQR UV O.P.. TUV YZA.
     XWVUTSS .O.LLH.HG .GGH .C.T...Q FE. DDE.JK.LHNO..S  W P QRS . U WXY.
       XW.U  TU.P. HN H FFFG..D..SRQP HGFF .E. F  M...   .X Q QQ...X.VVW .Z.
       .XV.U..      ....I.EE.  ..UTS. Q H.GG.F .QH K      .R RST W  WX.AAB
       .ZYY W VW R..     .D.     .QH.II HHGOH HHIJKLH. BB S.SSTUVWXYX.YZ..
       AZZ. .W. QP  .DDC. .VW .QHH JJII .III L     A  .TT.UVWXY.. Z.
        ..XW U.R O.  C B  .U. POWWLKKJJ .KJ JJ.H   Z WVU U..WXYZ.BAA.
        WVUTS.HH.   BA. ...T.Q...HLLK KLH.. KL     YXW.V.VW XYZ BB.
         ..TT.H.  .AZ. W.TSR.QP.H. LL H  .       . XX W .YY
     AZY.VUU.LL. .BAZYXWVU   .ON. ..        .     Y    .Z.ZA
     A..XW ....KL.....Z XWVVW.         .        AZ.      .A .
       XXY. J HH CBA..    .                               B
       ZYYY.  I.. DC.BCD
       A  .. IHGFED. CDEFGH.
          KJ..GFE.J D.FGH.
        .A   KK.. F.I .EF.
       AZ.     ..L GGH GF G.L.
        Y       LK.HH HG  H K
       XWWX.   H.JIIIHH..IJ..
       Y V.Y  NH.JJJ ..IJ.. P...
       Z. U PO.. .KK  JJKL.OP.Q.
          VUT Q..Q  L. LKK..HNDPQ..
         .W.TSRR. P H  H...  NOP RS...
         XWVU.S. PON. .    O.P.SS.V
           .V.. P.     . .QQ TTU
            W        Q.          R.UU X
             .                R..   .S  VW.
                               SS..VW     TTU W..
                                       .. AA.A .A
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Figure 1. Example of a part of an infinite cluster grown at criticality. The asterisk '*' is an arbitrary point in the cluster. Sites belonging to shells of successive chemical distance are designated by successive letters of the alphabet (starting with 'B'), and dots designate sites which were tested and found to be empty. The cluster was generated on a triangular lattice, obtained by taking points to lie on a square lattice and regarding sites along one of the diagonal directions as nearest neighbours.

at least L shells (hereafter referred to as 'large' clusters). We assume that the ensemble averages $\bar{B}(L)$, $\bar{S}(L)$, and $\bar{R}(L)$ all vary as algebraic powers of L. The chemical distance exponent $\tilde{\nu}$ is defined as

$$\overline{R^2} \sim L^{2\tilde{\nu}}. \tag{1}$$

Since

$$d\bar{S}/dL \sim \bar{B}, \quad \text{and} \quad \bar{S} \sim \bar{R}^{d_f}, \tag{2}$$

it follows that

$$\bar{S}(L) \sim L^{\tilde{\nu}d_f} \equiv L^{d_L} \tag{3}$$

and

$$\bar{B}(L) \sim L^{\tilde{\nu}d_f - 1} \equiv L^{d_L - 1}. \tag{4}$$

Thus, the exponents for $\bar{B}(L)$, $\bar{S}(L)$, and $\bar{R}(L)$ all are related to two parameters, $\tilde{\nu}$ and d_f. Note that, as a special case, these results hold for the infinite percolation cluster at criticality.

The previous discussion concerned the case of clusters whose size is at least L shells. If clusters which had terminated at lower shell number also are included in the statistical ensemble, averages can be calculated (Stauffer 1979, Ben-Avraham and Havlin 1982) by using the known distribution for percolation cluster sizes $P'(S) \sim S^{1-\tau}$.

For example, the mean $\langle S_L \rangle$ (on 'all' clusters) is given as

$$\langle S_L \rangle \sim \sum_S S(L) S^{1-\tau} \sim L^{(d_f - \beta/\nu)\tilde{\nu}}. \tag{5}$$

Similarly, we find

$$\langle B_L \rangle \sim L^{(d_f - \beta/\nu)\tilde{\nu} - 1}, \tag{6}$$

and

$$\langle R_L^2 \rangle \sim L^{(2-\beta/\nu)\tilde{\nu}}. \tag{7}$$

Another quantity of interest is P_L, the probability that a cluster will consist of at least L shells. This function is given as

$$P_L \sim R^{-\beta/\nu} \sim L^{-\beta\tilde{\nu}/\nu}, \tag{8}$$

which follows immediately from equations (1) and (3) and the fact that the probability of a cluster being at *least* of size S is given as $P_L \sim \Sigma_S^\infty P'(S) \sim S^{2-\tau}$. It is clear that quantities given in equations (5)–(7) can be obtained from analogous quantities in equations (1)–(3) simply by multiplying by the function given in equation (8), e.g., $\langle S_L \rangle = \bar{S}(L) \cdot P_L$.

By using similar arguments we can calculate the average chemical distance of the clusters in the entire ensemble when $p < p_c$. We observe that the average chemical distance $\langle L \rangle$ can be given as

$$\langle L \rangle \sim \sum^{S_\xi} L S^{1-\tau} \sim S_\xi^{2-\tau+1/d_f\tilde{\nu}} \sim |p - p_c|^{\beta-\nu/\tilde{\nu}}. \tag{9}$$

Using equation (9) and the relationship $\langle L \rangle = \bar{L} \cdot P_L$, we find that

$$\bar{L} \sim |p - p_c|^{-\nu/\tilde{\nu}} \tag{10}$$

where $\bar{L}$ is the 'correlation chemical length', i.e., the chemical distance associated with the correlation length ξ.

The exponents appearing in the above equations can be evaluated exactly for the case of percolation on the Cayley tree, which is believed to represent percolation clusters in six dimensions (de Gennes 1976). It can be shown that, because there is no correlation between the growing tips, in this case

$$\langle B_L \rangle = 1. \tag{11}$$

This result follows simply from the fact that, for an n-branched growth process at criticality, at each branch tip the probability of growing a new branch is $p = p_c = 1/n$ (Harris 1983) where $n = z - 1$, and z is the coordination number. Since for the Cayley tree it is known that $d_f = 4$, $\beta = 1$, and $\nu = \frac{1}{2}$ (Stauffer 1979), it follows from equation (6) that $\tilde{\nu} = \frac{1}{2}$.

Several other interesting facts can be concluded from this result. On the population of large clusters, those whose shell number is L or greater (including the infinite cluster), one finds from equations (1)–(4) the relationships

$$\overline{R^2} \sim L, \qquad \bar{S}(L) \sim L^2, \qquad \bar{B}(L) \sim L. \tag{12}$$

Because of the analogy between the Cayley tree and six-dimensional percolation, the first of these relationships indicates that the spatial extent R in 6D percolation clusters depends on the chemical distance L with the same exponent as a simple random walk.

We note that the expected number of sites in the Lth shell of an ensemble of large clusters, $B(L)$, varies linearly with L, which is consistent with the observation from equation (8) that $P_L \sim 1/L$ and the fact that $\langle B_L \rangle = 1$.

Another exponent of interest is the chemical distance diffusion exponent d_w^L, which characterises the average chemical distance traversed in t steps by a random walk on the infinite percolation cluster, namely,

$$L^{d_w^L} \sim t. \tag{13}$$

Using equation (1) and the spatial diffusion equation $R^{d_w} \sim t$, it follows that

$$d_w^L = \tilde{\nu} d_w. \tag{14}$$

Equation (14) is a general relationship which, when evaluated for a Cayley tree using the known value $d_w = 6$ and the result obtained above, $\tilde{\nu} = \frac{1}{2}$, yields the value $d_w^L = 3$. This conclusion is supported by the numerical data shown in figure 2. Another closely related quantity is the number of distinct sites visited by a random walk, which can be calculated from equations (3), (13), and (14) to be $S_t \sim t^{d_L/d_w^L} \sim t^{d_f/d_w} \equiv t^{\bar{d}/2}$. Thus, it is seen that the topological exponent $\bar{d}$ can be expressed as a ratio of topological exponents d_L/d_w^L, in addition to the usual ratio of spatial exponents d_f/d_w (Alexander and Orbach 1982).
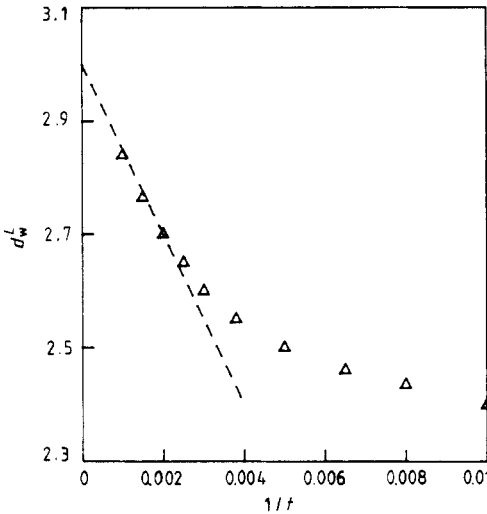
Figure 2. Exponent for diffusion on percolation clusters on a Cayley tree, plotted against $1/t$. The extrapolated limit as $t \to \infty$ seems to be $d_w^L = 3$. Diffusion was calculated exactly on individual clusters for which the chemical distance was at least $L = 200$, using methods similar to those applied previously for diffusion on exact fractals (Ben-Avraham and Havlin 1982). Results were averaged over a sample of 1000 configurations.

We also have evaluated these exponents for other dimensions. Numerical results for $\bar{B}(L)$ and $\langle B_L \rangle$ for 2D percolation clusters formed on a triangular lattice at criticality are shown in figure 3. From the slopes of these curves we find from equations (4) and (6) that $[\tilde{\nu} d_f - 1] = 0.64 \pm 0.02$ and $[(d_f - \beta/\nu)\tilde{\nu} - 1] = 0.57 \pm 0.03$. Using the known values (Stauffer 1979) $\beta = 0.14$, $\nu = \frac{4}{3}$, and $d_f = 1.89$, we obtain $\tilde{\nu} = 0.87 \pm 0.02$ from the data of each of the curves. It is interesting to compare these results with those of Alexandrowicz
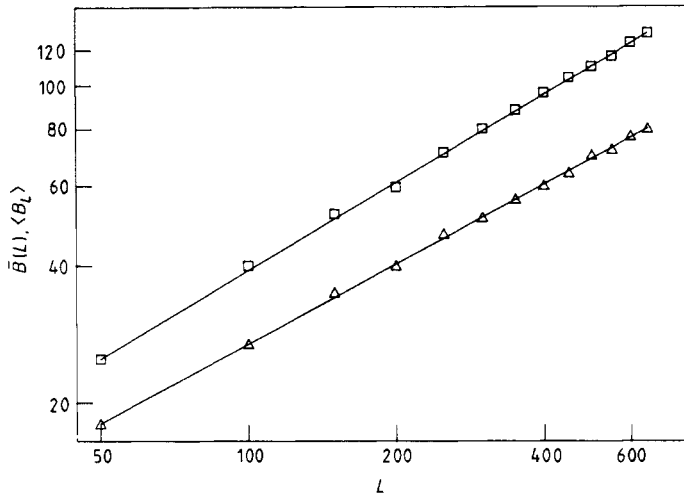
Figure 3. The average number of sites in the Lth shell $\bar{B}(L)$ ($\square$) and $\langle B_L \rangle$ ($\triangle$), as a function of chemical distance L. Results shown are for 1000 clusters.

(1980), who studied the growth of percolation clusters on hypercubic lattices. In the latter study, the clusters are grown as successive shells, one shell per unit 'time'; thus, the 'time' in Alexandrowicz' study corresponds to the chemical distance L used in our work. The three seemingly independent exponents γ_t, ν_t, and τ defined by Alexandrowicz (1980) correspond, respectively, to $(d_f - \beta/\nu)\tilde{\nu}$, $\tilde{\nu}$, and $\nu/\tilde{\nu}$ in our study (cf equations (5), (1), and (9)), and therefore now are seen to depend on only one apparently new exponent, $\tilde{\nu}$. Using data given by Alexandrowicz (1980) and known values for β and ν (Stauffer 1979, Stauffer *et al* 1982), we have computed $\tilde{\nu}$ for $d = 3$, 4 as well. Results are shown in table 1.

An interesting observation from the data in table 1 is the close correspondence between the values in the last two columns on the right. From those data we observe that, within the accuracy of the calculations, the following relationship between d_f, ν, and $\tilde{\nu}$ seems to be valid:

$$d_f = 1/\nu + 1/\tilde{\nu}. \tag{15}$$

The implication of the latter relationship is that $\tilde{\nu}$ is *not* an additional independent exponent but, rather, can be evaluated in terms of β and ν. This would be in accord

Table 1. Critical exponents for percolation clusters.

d	d_f	ν	β	$\tilde{\nu}$	$\tilde{\nu}d_f - 1$	$\tilde{\nu}/\nu$
1	1	∞[a]	0	1[b]	0	0
2	1.89	1.33[c]	0.14[c]	0.87 ± 0.02	0.64 ± 0.02[g]	0.65 ± 0.02
3	2.49	0.88[c]	0.45[c]	0.74 ± 0.02[d]	0.83 ± 0.02	0.83 ± 0.02
4	3.05	0.7[e]	0.65[f]	0.63 ± 0.04[d]	0.93 ± 0.04	0.91 ± 0.04
6	4	1/2[e]	1[e]	1/2	1	1

[a] Havlin *et al* (1983); [b] in $d = 1$, R and L are equivalent; [c] Stauffer *et al* (1982); [d] Alexandrowicz (1980); [e] Stauffer (1979); [f] deduced from values of γ and ν by Stauffer (1979); [g] this value is consistent with results obtained by Pike and Stanley (1981) who found $d_L = 1.64 \pm 0.02$.

with the generally accepted belief that all static geometrical properties of percolation clusters can be described in terms of two independent exponents. It follows from equations (1), (4), and (15) that $B \sim R^{1/\nu}$, which suggests that $1/\nu$ is a fractal index. Because $1/\tilde{\nu}$ also is a fractal index, it seems that d_f is the sum of two fractal indices. A simple theoretical assumption which would yield equation (15) is that $B \sim 1/|p - p_c|$ for all d; from equations (1) and (4) and the relation $\bar{R} \sim |p - p_c|^{-\nu}$ one obtains $\bar{B} \sim |p - p_c|^{-\nu/\tilde{\nu}(\tilde{\nu}d_f - 1)}$, so that the aforementioned assumption immediately implies the relationship for the exponents given in equation (15).

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